THE ROLE OF PRIMES OF GOOD REDUCTION IN THE BRAUER–MANIN OBSTRUCTION

Margherita Pagano

Leiden University

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Introduction

Let k be a number field and Ω_k be the set of places of k. Let X be a smooth, proper, geometrically integral k-variety. We are interested in understanding the set $X(k)$ of k-points on X.
For every $\nu \in \Omega_k$ we have

$$X(k) \hookrightarrow X(k_\nu).$$

Hence,

$$X(k) \hookrightarrow \prod_{\nu \in \Omega_k} X(k_\nu).$$

Question: what does the image of $X(k)$ in $\prod_{\nu \in \Omega_k} X(k_\nu)$ look like?

Weak approximation

Definition

We say that X satisfies *weak approximation* if the image

$$X(k) \hookrightarrow \prod_{\nu \in \Omega_k} X(k_\nu)$$

is dense.

(Counter)example

1. The projective spaces $\mathbb{P}_k^n$ satisfy weak approximation.
2. $Q \subseteq \mathbb{P}_k^n$ smooth projective quadric satisfies weak approximation.
3. Selmer's example: the projective variety defined by the equation $3x^3 + 4y^3 + 5z^3 = 0$ has a $\mathbb{Q}_\nu$-point for every place $\nu \in \Omega_\mathbb{Q}$ but does not admit a rational point.

Manin, 1970

Manin has shown that it is possible to use the Brauer group to build a closed subset $\left(\prod_{\nu\in\Omega_k} X(k_\nu)\right)^{\mathrm{Br}} \subseteq \prod_{\nu\in\Omega_k} X(k_\nu)$ that contains $X(k)$. Hence,

$$X(k) \subseteq \overline{X(k)} \subseteq \left(\prod_{\nu\in\Omega_k} X(k_\nu)\right)^{\mathrm{Br}} \subseteq \prod_{\nu\in\Omega_k} X(k_\nu).$$

Idea: $\left(\prod_{\nu\in\Omega_k} X(k_\nu)\right)^{\mathrm{Br}}$ is easier to describe than $\overline{X(k)}$.

Brauer–Manin obstruction

We say that there is a *Brauer–Manin obstruction to weak approximation* on X if $\left(\prod_{\nu\in\Omega_k} X(k_\nu)\right)^{\mathrm{Br}} \subsetneq \prod_{\nu\in\Omega_k} X(k_\nu)$.

Construction of the B-M set: the evaluation map

Let $\mathcal{A} \in \mathrm{Br}(X)$, then for every place $\nu \in \Omega_k$ we get an induced map $\mathrm{ev}_{\mathcal{A}}$, called the evaluation map:

$$\mathrm{ev}_{\mathcal{A}} : X(k_\nu) \to \mathbb{Q}/\mathbb{Z}.$$

The Brauer–Manin set, $\left(\prod_{\nu\in\Omega_k} X(k_\nu)\right)^{\mathrm{Br}}$, is defined as:

$$\left\{(x_\nu) \in \prod_{\nu\in\Omega_k} X(k_\nu) \mid \sum_{\nu\in\Omega_k} \mathrm{ev}_{\mathcal{A}}(x_\nu) = 0,\ \forall \mathcal{A} \in \mathrm{Br}(X)\right\}.$$

Places involved in the Brauer–Manin obstruction

Let $\mathcal{A} \in \mathrm{Br}(X)$ and $\omega \in \Omega_k$ be such that

$$\mathrm{ev}_{\mathcal{A}} : X(k_\omega) \to \mathbb{Q}/\mathbb{Z}$$

is non-constant, then there is a Brauer-Manin obstruction to weak approximation (and we say that the place ω plays a role in the Brauer–Manin obstruction to weak approximation).

Proof.

Pick

$$(x_\nu), (y_\nu) \in \prod_\nu X(k_\nu) \text{ such that } \begin{cases} x_\nu = y_\nu \text{ for all } \nu \neq \omega, \\ \mathrm{ev}_{\mathcal{A}}(x_\omega) \neq \mathrm{ev}_{\mathcal{A}}(y_\omega). \end{cases}$$

Then

$$\sum_\nu \mathrm{ev}_{\mathcal{A}}(x_\nu) \neq \sum_\nu \mathrm{ev}_{\mathcal{A}}(y_\nu).$$

□

Swinnerton-Dyer's question

Let X be such that $\mathrm{Pic}(X \times_k \overline{k})$ is torsion-free and finitely generated. Let $S \subseteq \Omega_k$ be a finite set of places consisting of the archimedean places and the places of bad reduction for X.

Question

Is it true that the only places that play a role in the Brauer–Manin obstruction to weak approximation on X are the places of bad reduction and the archimedean places?

Previous works

Theorem [Colliot-Thélène–Skorobogatov]

Assume that $\omega \in \Omega_k$ is a non archimedean place of good reduction whose residue characterstic does not divide the order of $\mathrm{Br}(X)/\mathrm{Br}_1(X)$. Then for every $\mathcal{A} \in \mathrm{Br}(X)$

$$\mathrm{ev}_{\mathcal{A}} : X(k_\omega) \to \mathbb{Q}/\mathbb{Z}$$

is constant.

Theorem [Bright-Newton 2020]

Assume that $\mathrm{H}^0(X, \Omega^2_X) \neq 0$. Let $\mathfrak{p}$ be a prime at which X has good ordinary reduction. Then there exist a finite extension L/k such that there is an element $\mathcal{A} \in \mathrm{Br}(X_L)$ whose evaluation map

$$\mathrm{ev}_{\mathcal{A}} : X(L_{\mathfrak{p}'}) \to \mathrm{Br}(L_{\mathfrak{p}'})$$

is non-constant.

K3 surfaces

Definition

A K3 *surface* over a number field k is a smooth, geometrically integral, 2-dimensional k-variety such that

$$\mathrm{H}^1(X, \mathcal{O}_X) = 0 \quad \text{and} \quad \omega_X \simeq \mathcal{O}_X.$$

- The Picard group $\mathrm{Pic}(X \times_k \overline{k})$ is always torsion-free and finitely generated (hypothesis in S-D question ✓).
- $\mathrm{H}^0(X, \Omega^2_X) \neq 0$ (hypothesis in B–N theorem ✓).

Example

Let $X \subseteq \mathbb{P}^3_{\mathbb{Q}}$ be the K3 surface defined by the equation

$$x^3y + y^3z + z^3w + w^3x + xyzw = 0.$$

X has good ordinary reduction at the prime 2 and the following theorem holds true.

Theorem (P.)

The class of the quaternion algebra

$$\mathcal{A} = \left(\frac{z^3 + w^2x + xyz}{x^3}, -\frac{z}{x} \right) \in \mathrm{Br}(\mathbb{Q}(X))$$

defines an element in $\mathrm{Br}(X)$. *The evaluation map* $\mathrm{ev}_{\mathcal{A}} : X(\mathbb{Q}_2) \to \mathrm{Br}(\mathbb{Q}_2)$ *is non-constant.*

Work in progress

Theorem (Bright–Newton 2020)

Let X/k be a K3 surface and $\mathfrak{p}$ be a prime of good reduction for X. If $e_{\mathfrak{p}} < p - 1$ then $\mathfrak{p}$ does not play a role in the Brauer–Manin obstruction to weak approximation.

Theorem (P.)

Let X/k be a K3 surface and $\mathfrak{p}$ be a prime of good ordinary reduction for X. If $(p-1) \nmid e_{\mathfrak{p}}$ then $\mathfrak{p}$ does not play a role in the Brauer–Manin obstruction to weak approximation.

Theorem (P.)

Let X/k be a K3 surface and $\mathfrak{p}$ be a prime of good non-ordinary reduction for X. If $e_{\mathfrak{p}} \leq p - 1$ then $\mathfrak{p}$ does not play a role in the Brauer–Manin obstruction to weak approximation.

Thank you for the attention!